Computational Topology

Lecturers: Francis Lazarus and Arnaud de Mesmay

Where and when: See timetables.

Prerequisites: Nothing strictly required. Basics in graph theory and algorithms will help, as well as a certain taste for mathematics.

Objective: Computational topology is primarily concerned with the development of efficient algorithms for solving topological problems. This course is an introduction to the main tools and concepts in the field. While topology is an old and mature mathematical field, the study of its effective aspects has only started to flourish in the last decades. We will start with graph theory using planar and surface-embedded graphs to introduce fundamental topological notions as we progress. We then increase the dimension progressively and finish with persistence theory, a blooming topological tool in the analysis of big data.

Tentative plan:

1. Planar graphs: Jordan theorem, Euler relation, forbidden minors, Tutte embedding.
2. Classification of surfaces and surface-embedded graphs, Euler characteristic, canonical systems of loops. Homotopy and homology, greedy generators. Homotopy test, intersection numbers.
3. Elementary 3-dimensional topology: triangulations, knots, knot diagrams, Reidemeister moves. Normal surfaces and algorithm to test knot triviality.
4. Simplicial complexes and limits of computational topology in high dimensions: Undecidability of the word problem, and therefore undecidability of computations in the fundamental group of a 2-complex, and undecidability of 4-manifold homeomorphism or 5-sphere recognition.
5. Persistent homology and applications to topological data analysis.

Course notes: ,

- Introductory course
- Planar graphs, — **Exercise sheet 1**
  - **Exercise sheet 2**
- Surfaces, — **Exercise sheet 3**, **(Partial) solution**
  - **Exercise sheet 4 is due October 19 and will be rated**, **Solution**
  - **Exercise sheet 5**
- Homotopy test, — **Exercise sheet 6**
- Minimum weight Bases — **Exercise sheet 7**
- Homology computation — **Exercise sheet 8**
  - **Exercise sheet 9 is due November 30 and will be rated**
- Knots and 3-d computational topology — **Exercise sheet 10**

Exercise sheet 11

Exercise sheet 12 is due January 12 and will be rated

- Undecidability in topology
- Persistent homology

Suggested reading:

- John Stillwell. Classical Topology and Combinatorial Group Theory. Springer, 1995.
- Jonathan L. Gross and Thomas W. Tucker. Topological Graph Theory. Dover Publications, 2001.
- Bojan Mohar and Carsten Thomassen. Graphs on Surfaces. Johns Hopkins University Press, 2001.
- Joel Hass, Jeffrey C. Lagarias, Nicholas Pippenger. The Computational Complexity of Knot and Link Problems. Journal of the ACM (JACM), 46(2), 185-211.
- Herbert Edelsbrunner and John Harer. Persistent homology-a survey. Contemporary mathematics, 2008.

There are also good lecture notes available for a few other courses on Computational Topology. In particular, we recommend the ones of Éric Colin de Verdière, Jeff Erickson and Herbert Edelsbrunner. They do not necessarily cover the same topics, but might provide interesting perspectives on different facets of the subject.

Validation: Homework, and work (written report and oral presentation) on a research article. The rules are: three homeworks will be rated and we will consider the average HS of the two best scores. The final score will be obtained by the formula (HS+OS)/2, where OS is the score for the oral presentation.

Oral presentation: You have to select one project in the list below. We recommend that you choose a partner and work by pairs on a project. Each project should be chosen by at most one pair. Try to avoid conflict by discussing your choice with the whole class. The presentation should last **25** mn followed by questions and a short discussion. You may use your own laptop or bring your presentation on a usb key. In this latter case, we ask you to use a pdf file. Each member of a pair should speak approximately the same amount of time. The rough idea is to present the main ideas in the chosen article(s), give some details and say one proof that you find worth be presented. We do not require a written report for the project, but **we will ask for an electronic copy of your slides**. Some papers are only accessible via your institution. We nonetheless link each paper to a public version so that you get an idea of the content. When indicated, you should download the final version in the end. Please, send us an email if you have any problem to download a paper.

1. Minimum cycle and homology bases of surface embedded graphs (Erin W. Chambers, Glencora Borradaile, Kyle Fox and Amir Nayyeri)
2. The computational complexity of knot genus and spanning area (Ian Agol, Joel Hass and William P. Thurston)
3. Linkless and flat embeddings in 3 space and the unknot problem (Ken-Ichi Kawarabayashi, Stephan Kreutzer and Bojan Mohar). Final version.
4. Shellability is NP-complete (Xavier Goaoc, Pavel Paták, Zuzana Patáková, Martin Tancer and Uli Wagner)
5. Testing Isotopy of Graphs on Surfaces (Eric Colin de Verdière and Arnaud de Mesmay)
6. Optimally cutting a surface into a disk (Jeff Erickson and Sariel Har-Peled)
7. Homology flows, cohomology cuts (Erin W. Chambers, Jeff Erickson and Amir Nayyeri)
8. Algorithms for normal curves on surfaces (Marcus Schaefer, Eric Sedgwick and Daniel Stefankovic)
9. Optimal Homologous Cycles, Total Unimodularity, and Linear Programming (Tamal K. Dey, Anil N. Hirani and Bala Krishnamoorty)
10. Hardness of embedding simplicial complexes into R^d (Jirí Matousek, Martin Tancer and Uli Wagner)
11. A Proof of the Orbit Conjecture for Flipping Edge-Labelled Triangulations (Anna Lubiw, Zuzana Masárová and Uli Wagner)
12. Computing the Geometric Intersection Number of Curves (Vincent Despré and Francis Lazarus), version détaillée
13. Quantifying Homology Classes (Chen, C. & Freedman, D.)

14. Covering nearly surface-embedded graphs with a fixed number of balls (Glencora Borradaile and Erin Wolf Chambers)
15. Proximity of Persistence Modules and their Diagrams (Chazal, F.; Cohen-Steiner, D.; Glisse, M.; Guibas, L. & Oudot, S.) and
    Stability of Persistence Diagrams (Cohen-Steiner, D.; Edelsbrunner, H. & Harer, J.) Final version

Advanced topics: the following articles are significantly harder, pick them only if you are interested and/or you want to impress us.

16. Computing all maps into a sphere (Martin Cadek, Marek Krcal, Jiri Matousek, Francis Sergeraert, Lukas Vokrinek, Uli Wagner)
17. Embeddability in the 3-sphere is decidable (Jiri Matousek, Eric Sedgwick, Martin Tancer and Uli Wagner)
