Catching Polygons

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Abstract

Consider an arrangement of k lines intersecting the unit square. There is some minimum scaling factor so that any placement of a rectangle with aspect ratio $1 \times p$ with $p \geq 1$ must non-transversely intersect some portion of the arrangment or unit square. Assuming the lines of the arrangement are axis-aligned, we show the optimal arrangement depends on the aspect ratio of the rectangle. In particular, the optimal arranment is either evenly spaced parallel lines or an evenly spaced grid of lines. We present the precise aspect ratios of rectangles for which each of the two nets are optimal.

1   Introduction

During the open problem session of CCCG20, Joseph O'Rourke suggested the following problem. Consider an arrangement of k lines, all of which intersect the unit square. For a fixed polygon P, there is some minimum scale factor $c > 0$ such that the scaled polygon cP cannot be embedded in the unit square without intersecting any of the k lines of the arrangement non-transversely (allowing translation and rotation). That is, the scaled polygon will 'just touch' at least one line of the arrangement. Can we compute the minimum such c over all possible arrangements of this type? Can we describe an arrangement that realizes this minimum? See Figure 1 for an example.

This problem can be described using an analogy to catching fish. In this analogy, the polygon is a fish that lives in the unit square. The fish can vary in size but always has the same shape. The goal is to catch fish, that is, create a net so that the largest fish that can get through any hole in the net is as small as possible.
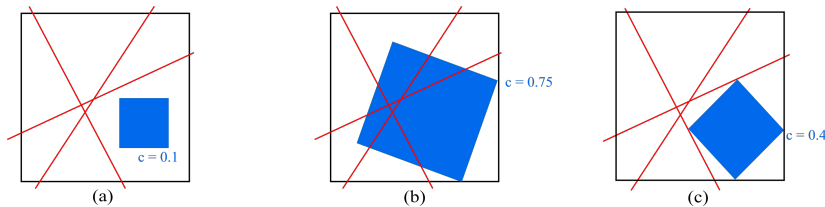

Figure 1: (a) If c is small, the polygon can avoid the net. (b) If c is large, any rotation and translation of the polygon is caught in the net. (c) The minimum c such that the polygon is caught in the net.

Here, we consider the special case where the lines are axis-aligned and P is a rectangle (a lemma towards justifying the setting of axis-aligned lines is given in Appendix B) We observe that, depending on the aspect ratio of the rectangle being considered, the optimal arrangement is either evenly spaced parallel lines or a grid of lines. See Figure 2 for an example. O'Rourke asked, *for what aspect ratios of rectangles is each of the two nets optimal?* In this work, we answer this question precisely. We are only aware of one work that directly considers this problem [1], however the author has a different interpretation of the problem. The problem we consider is closely related to the laser based localization problems considered in [3]. Using the results from [2], given any net and aspect ratio for the fish one can compute the optimal scale factor.

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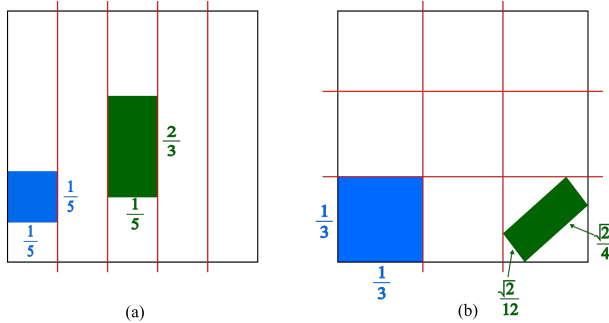
Figure 2: (a) For a square the evenly spaced vertical lines are optimal but for a rectangle with a (1×3) aspect ratio evenly spaced vertical lines is not optimal. (b) For a square the evenly spaced grid is not optimal but for a (1×3) rectangle the evenly spaced grid is optimal.

2   Rectangular Fish in Rectangular Nets

When P is a rectangle and the k lines are axis-aligned, the problem of calculating the optimal scale factor c reduces to finding the largest rectangle inscribed inside of another rectangle. Inscribing rectangles inside rectangles has been studied in [4, 5, 6]. In this section, we construct a curve that describes the optimal scale factor c for rectangular fish in rectangular nets for all spect ratios of the inscribed rectangle.

We fix the aspect ratio of the hole in the net to be $1 \times n$ with $n \geq 1$. Let the aspect ratio of the fish be $1 \times p$ with $p \geq 1$. We express the optimal scale factor c as a function of the aspect ratio p. We denote this curve by $\mathcal{C}_n(p)$. See Figure 3 for an example.
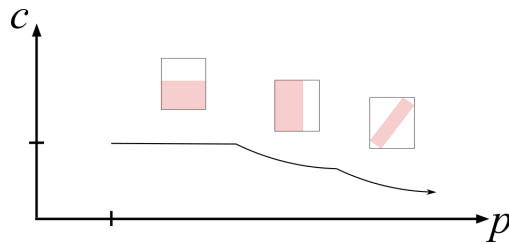

Figure 3: The inscribing curve for some $n \in \mathbb{R}^{\geq 1}$. The x-axis is the aspect ration of the fish and the y-axis is the optimal scale factor.

The curve $\mathcal{C}_n(p)$ consists of three parts. For small $p \leq n$, placing the shorter side of the fish parallel to the shorter side of the net is optimal and $c = 1$. For medium size p values relative to n, the value of c is limited by the height of the net and the longer side of the fish is scaled to equal the longer side of the net. This gives $pc = n$ or $c = \frac{n}{p}$.

Figure 4: A diagonally inscribed rectangle.
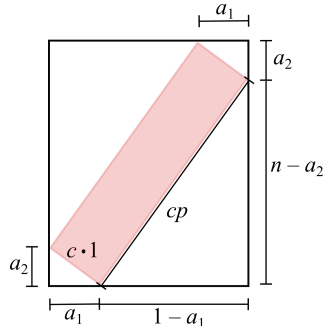
For large p relative to n, placing the fish diagonally is optimal. Using the variables indicated in Figure 4, we have the following three equations

$$\frac{a_1}{a_2} = \frac{n - a_2}{1 - a_1} \tag{1}$$

$$a_1^2 + a_2^2 = c^2 \tag{2}$$

$$(1 - a_1)^2 + (n - a_2)^2 = (cp)^2 \tag{3}$$

along with the natural constraints of the problem, $0 < c$, $0 < a_1 < 1$, $0 < a_2 < n$. Equation 1 is due to all triangles being similar. Equation 2 and Equation 3 are applications of Pythagorean's theorem.

2

Solving for c gives

$$c = \sqrt{\left(\frac{n-p}{1-p^2}\right)^2 (p^2+1) - 2\left(\frac{n-p}{1-p^2}\right)p + 1}.$$

Therefore, we are able to give an explicit formula for $\mathcal{C}_n(p)$ for a given n, namely,

$$\mathcal{C}_n(p) = \begin{cases} 1 & 1 \le p \le n \\ \frac{n}{p} & n < p \le w_n \\ \sqrt{\left(\frac{n-p}{1-p^2}\right)^2 (p^2+1) - 2\left(\frac{n-p}{1-p^2}\right)p + 1} & w_n < p \end{cases}$$

where w_n is the solution to $\sqrt{\left(\frac{n-p}{1-p^2}\right)^2 (p^2+1) - 2\left(\frac{n-p}{1-p^2}\right)p + 1} = \frac{n}{p}$ for $1 < n < p$. The value w_n represents the scale factor where the vertically inscribed rectangle has the same scale factor as the diagonally inscribed rectangle.

When k is even, the grid with $\frac{k}{2}$ horizontal lines and $\frac{k}{2}$ vertical lines has square holes with side length $\frac{1}{\frac{k}{2}+1}$. The arrangement with k vertical and 0 horizontal lines has rectangular holes with dimension $1 \times \frac{1}{k+1}$. The curves $\frac{1}{\frac{k}{2}+1}\mathcal{C}_1(p)$ and $\frac{1}{k+1}\mathcal{C}_{k+1}(p)$ intersect at $p = \frac{k+1}{\frac{k}{2}+1}$. For even values of k, we define the *base curve* to be

$$B_k(p) = \min\left\{\frac{1}{k+1}\mathcal{C}_{k+1}(p), \frac{1}{\frac{k}{2}+1}\mathcal{C}_1(p)\right\}$$

The case for odd values of k is similar and included in Appendix A.

3 Optimality Results

In this section, we show that for any axis-aligned net $\mathcal{N}$ with an even number of lines and any aspect ratio of the fish p, the base curve has a scale factor that is less than or equal to the scale factor of $\mathcal{N}$. Let $\mathcal{N}(k,0)$ denote the net with k evenly spaced parallel lines and let $\mathcal{N}(\frac{k}{2}, \frac{k}{2})$ denote the net with $\frac{k}{2}$ evenly spaced vertical lines and $\frac{k}{2}$ evenly spaced horizontal lines. The rectangle aspect ratio where the base curve switches from $\mathcal{N}(k,0)$ to $\mathcal{N}(\frac{k}{2}, \frac{k}{2})$ is $p = \frac{k+1}{\frac{k}{2}+1}$. We now state our main theorem:

Theorem 1 (Regular is Optimal). *For k even, the optimal axis aligned net for rectangular polygons is $\mathcal{N}(k,0)$ for aspect ratio $p \le \frac{k+1}{\frac{k}{2}+1}$ and $\mathcal{N}(\frac{k}{2}, \frac{k}{2})$ for aspect ratio $p \ge \frac{k+1}{\frac{k}{2}+1}$.*

Proof. First, notice that regularly spaced lines give a smaller scale factor than irregularly spaced lines. This is because a rectangular hole generated by regular spacing has height and width equal to the average. A rectangular hole generated by irregular spacing has a hole with height and width greater than or equal to the average.

Consider any axis aligned net over the square, let v be the number of vertical lines and h be the number of horizontal lines call the net $\mathcal{N}(v,h)$. Notice $v + h = k$. If $v = n$ then we have a $\frac{k}{2} \times \frac{k}{2}$ net and if $n = 0$ (or $v = 0$), then we have k parallel lines. The base curve is defined to be the minimum scale factor of $\mathcal{N}(\frac{k}{2}, \frac{k}{2})$ and $\mathcal{N}(k,0)$. Thus, the scale factor of the base curve is less than or equal to either of these nets.

Consider any other v and h. Without loss of generality, assume $h < v$ we have $0 < h < \frac{k}{2} < v < k$. There are $(v+1)(h+1)$ holes in the net and the average size of a hole is $\frac{1}{v+1} \times \frac{1}{h+1}$. There exists one hole at least as big as the average, so, we have a hole at least as big as the hole with aspect ratio $\frac{v+1}{h+1}$ scaled so the smaller side has length $\frac{1}{v+1}$. The optimal scale factor of a rectangle with aspect ratio p that fits inside this hole is given by $\frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p)$. We compare $\frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p)$ to $B_k(p)$. Both $\frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p)$ and $B_k(p)$ are piecewise functions, we directly examine all $p \geq 1$ to show $B_k(p) \leq \frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p)$, see Figure 5 for intuition.

For $1 \leq p \leq \frac{k+1}{\frac{k}{2}+1}$, we have $B_k(p) = \frac{1}{k+1}$ and $\frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p) = \frac{1}{v+1}$. since $v < k$, we have $\frac{1}{k+1} < \frac{1}{v+1}$. For $\frac{k+1}{\frac{k}{2}+1} \leq p \leq \frac{v+1}{h+1}$, $B_k(p)$ decreases and $\frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p) = \frac{1}{k+1}$ is constant.

For $\frac{v+1}{h+1} < p \leq w_{\frac{v+1}{h+1}}$, $B_k(p) = \min\left\{\frac{1}{k+1}\mathcal{C}_{k+1}(p), \frac{1}{\frac{k}{2}+1}\mathcal{C}_1(p)\right\} \leq \left(\frac{1}{\frac{k}{2}+1}\right)\left(\frac{1}{p}\right)$ and $\frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p) = \frac{1}{v+1}\left(\frac{v+1}{h+1}\right)\frac{1}{p}$. Since $h \leq \frac{k}{2}$, we have

$$\left(\frac{1}{\frac{k}{2}+1}\right)\left(\frac{1}{p}\right) < \frac{1}{h+1}\left(\frac{1}{p}\right) = \frac{1}{v+1}\left(\frac{v+1}{h+1}\right)\frac{1}{p}.$$

For $p \geq w_{\frac{v+1}{h+1}}$ the interior rectangle is placed diagonally. Let c' be the scale factor value of the rectangle placed diagonally in $\mathcal{N}(\frac{k}{2}, \frac{k}{2})$. Consider the length of the rectangle, with shorter side equal to c', placed diagonally in a rectangle with sides $\frac{1}{v+1} \times \frac{1}{h+1}$. Let a_1 and a_2 be the legs of the small right triangle formed by c', the squared length of this inscribed rectangle is $\ell^2(v, h, a_1, a_2) = (\frac{1}{v+1} - a_2)^2 + (\frac{1}{h+1} - a_1)^2$. The minimum of this function along the constraints Equation 1, Equation 2, and $v + h = k$ occurs when $v = h$. We omit the details, but this can be done with Lagrange multipliers. So, when $v \neq h$ we can fit the diagonal rectangle of $\mathcal{N}(\frac{k}{2}, \frac{k}{2})$ inside the diagonal of the rectangle with dimensions $\frac{1}{v+1} \times \frac{1}{h+1}$ so the optimal scale factor must be larger. □
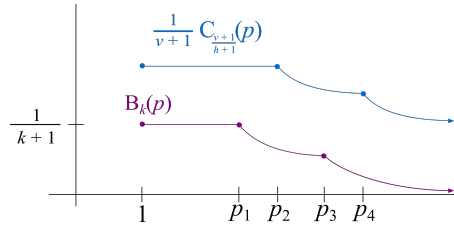

Figure 5: The curves $B_k(p)$ and $\frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p)$. Here, $p_1 = \frac{k+1}{\frac{k}{2}+1}$, $p_2 = \frac{v+1}{h+1}$, $p_3 = \sqrt{2} + 1$, and $p_4 = w_{\frac{v+1}{h+1}}$.

4 Discussion

In this work, we showed that for axis-aligned nets with k lines and rectangular fish with aspect ratio $1 \times p$, the optimal net is evenly spaced parallel lines for $p \leq \frac{k+1}{\frac{k}{2}+1}$ and a grid of $\frac{k}{2} \times \frac{k}{2}$ evenly spaced lines for $p \geq \frac{k+1}{\frac{k}{2}+1}$. As far as we know, the problem is still open for non axis-aligned nets and rectangular fish. Our hope is that someone can show, that for any net, one can construct an axis-aligned net that does change the optimal scale factor. Non-rectangular fish would also be interesting to explore.

4

References

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A   An odd number of lines

In this section, we prove that, when k is odd, the net $\mathcal{N}(k,0)$ is optimal for $1 \leq p \leq \frac{(k+1)\lfloor\frac{k}{2}\rfloor}{\lceil\frac{k}{2}\rceil^2}$ and the net $\mathcal{N}(\lceil\frac{k}{2}\rceil, \lfloor\frac{k}{2}\rfloor)$ is optimal for $p \geq \frac{(k+1)\lfloor\frac{k}{2}\rfloor}{\lceil\frac{k}{2}\rceil^2}$. The net $\mathcal{N}(\lceil\frac{k}{2}\rceil, \lfloor\frac{k}{2}\rfloor)$ has $\frac{1}{\lceil\frac{k}{2}\rceil} \times \frac{1}{\lfloor\frac{k}{2}\rfloor}$ holes and the net $\mathcal{N}(k,0)$ has $1 \times \frac{1}{k+1}$ holes. We consider the curves $\frac{1}{\lceil\frac{k}{2}\rceil}\mathcal{C}_{\lceil\frac{k}{2}\rceil \atop \lfloor\frac{k}{2}\rfloor}(p)$ and $\frac{1}{k+1}\mathcal{C}_{k+1}(p)$. See Figure 6. These curves intersect at $p = \frac{(k+1)\lfloor\frac{k}{2}\rfloor}{\lceil\frac{k}{2}\rceil^2}$. We define the *base curve* for k odd to be

$$D_k(p) = \min\left\{\frac{1}{k+1}\mathcal{C}_{k+1}(p), \frac{1}{\lceil\frac{k}{2}\rceil}\mathcal{C}_{\lceil\frac{k}{2}\rceil \atop \lfloor\frac{k}{2}\rfloor}(p)\right\}.$$
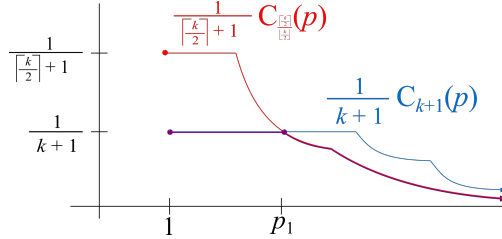

Figure 6: The minimum curve for k odd, with $p_1 = \frac{(k+1)\lfloor\frac{k}{2}\rfloor}{\lceil\frac{k}{2}\rceil^2}$.

Theorem 2 (Regular is Optimal Odd). *For k odd, the optimal axis aligned net for rectangular polygons is $\mathcal{N}(k,0)$ for aspect ratio $p \leq \frac{(k+1)\lfloor\frac{k}{2}\rfloor}{\lceil\frac{k}{2}\rceil^2}$ and $\mathcal{N}(\lceil\frac{k}{2}\rceil, \lfloor\frac{k}{2}\rfloor)$ for $p \geq \frac{(k+1)\lfloor\frac{k}{2}\rfloor}{\lceil\frac{k}{2}\rceil^2}$.*

Proof. Consider any axis aligned net over the square, let v be the number of vertical lines and h be the number of horizontal lines call the net $\mathcal{N}(v,h)$. Notice $v + h = k$.

Recall evenly spaced lines give a smaller scale factor than irregularly spaced lines. If $v = \lceil\frac{k}{2}\rceil$ and $\lfloor\frac{k}{2}\rfloor = n$ then we have a $\lceil\frac{k}{2}\rceil \times \lfloor\frac{k}{2}\rfloor$ net. If $n = 0$ (or $v = 0$), then we have k parallel lines. The base curve is defined to be the minimum scale factor of $\mathcal{N}(k,0)$ and $\mathcal{N}(\lceil\frac{k}{2}\rceil, \lfloor\frac{k}{2}\rfloor)$ so the base curve has scale factor less than or equal to $\mathcal{N}(k,0)$.

Consider any other v and h. Without loss of generality assume $h < v$ we have $0 < h < \lfloor\frac{k}{2}\rfloor < \lceil\frac{k}{2}\rceil < v < k$

There are $(v+1)(h+1)$ holes in the net and the average size of a hole is $\frac{1}{v+1} \times \frac{1}{h+1}$. There exits one hole at least as big as the average, that is, with width at least $\frac{1}{v+1}$ and height at least $\frac{1}{h+1}$. So we have a hole with aspect ratio $\frac{v+1}{h+1}$ scaled so the smaller side has length $\frac{1}{v+1}$. The maximum scale factor of a rectangle

with aspect ratio p that fits inside this hole is given by $\frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p)$. We compare $\frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p)$ to $D_k(p)$. Both $\frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p)$ and $D_k(p)$ are piecewise functions, we directly examine all $p \geq 1$ to show $D_k(p) < \frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p)$.

For $1 \leq p \leq \frac{(k+1)\lfloor\frac{k}{2}\rfloor}{\lceil\frac{k}{2}\rceil^2}$, we have $D_k(p) = \frac{1}{k+1}$ and $\frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p) = \frac{1}{v+1}$. since $v < k$, we have $\frac{1}{k+1} < \frac{1}{v+1}$. Then, for $\frac{(k+1)\lfloor\frac{k}{2}\rfloor}{\lceil\frac{k}{2}\rceil^2} \leq p \leq \frac{v+1}{h+1}$, $D_k(p)$ decreases and $\frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p) = \frac{1}{k+1}$ is constant.

For $\frac{v+1}{h+1} < p \leq w_{\frac{v+1}{h+1}}$, $D_k(p) = \min\left\{\frac{1}{k+1}\mathcal{C}_{k+1}(p), \frac{1}{\lceil\frac{k}{2}\rceil}\mathcal{C}_{\frac{\lceil\frac{k}{2}\rceil}{\lfloor\frac{k}{2}\rfloor}}(p)\right\} \leq \frac{1}{\lceil\frac{k}{2}\rceil}\left(\frac{1}{p}\right)$ and

$\frac{1}{v+1}\mathcal{C}_{\frac{v+1}{h+1}}(p) = \frac{1}{v+1}\left(\frac{v+1}{h+1}\right)\frac{1}{p}$. Since $h \leq \lfloor\frac{k}{2}\rfloor$, we have

$$\left(\frac{1}{\lceil\frac{k}{2}\rceil}\right)\left(\frac{1}{p}\right) < \frac{1}{h+1}\frac{1}{p} = \frac{1}{v+1}\left(\frac{v+1}{h+1}\right)\frac{1}{p}.$$

For $p \geq w_{\frac{v+1}{h+1}}$, we again solve the same constrained optimization problem as in the even case. The minimum occurs when $v = h$. This is a global minimum, so in the odd case the minimum occurs when me make v as close to h as possible. So, when $h < \lfloor\frac{k}{2}\rfloor < \lceil\frac{k}{2}\rceil < v$ we can fit the diagonal rectangle of $\mathcal{N}(\lceil\frac{k}{2}\rceil, \lfloor\frac{k}{2}\rfloor)$ inside the diagonal of the rectangle with dimensions $\frac{1}{v+1} \times \frac{1}{h+1}$ and the optimal scale factor must be larger.

$\square$

B   Towards Axis-Aligned Net Optimality

In this appendix, we show that, for square fish, evenly spaced axis-aligned vertical lines are generally a local optimum. This is a first step toward our conjecture that axis-aligned lines are a global optimum for rectangular fish.

Lemma 1. *Let P be a fish with aspect ration $p = 1$, i.e., a square. Then evenly spaced axis-aligned vertical lines are a local optimum when the number of lines is $k > 2$.*

Proof. Denote the k lines from left to right by $\ell_1, \ell_2, \ldots, \ell_k$. Assume, towards a contradiction, that, given some small $\epsilon > 0$, there are small shift and pivot values for each line that describe the translation of lines to an arrangement that results in a lower overall scaling factor c. Specifically, let these values be denoted $s_1, s_2, \ldots, s_k$ and $p_1, p_2, \ldots, p_k$, respectively, where $s_i, p_i \in [0, \epsilon]$, and at least one of these values s_i or p_i are nonzero. Notice that, regardless of the pivot value, shifting any line decreases the maximum fish size for one neighboring face, but increases it for the other neighboring face, leading to a higher c-value overall. Thus, we must have $s_1 = s_2 = \ldots = s_k = 0$. This means we must consider the effect of pivot values on unshifted lines. Notice that pivoting a single line that is adjacent to a vertical (un-pivoted) line will increase the maximum fish size for the face between them. Since the lines p_1 and p_k are always adjacent to the vertical edges of the bounding square, we must have $p_1 = p_k = 0$, otherwise the leftmost and rightmost faces would cause the arrangement to have a higher c-value. But then we must also have $p_2 = p_{k-1} = 0$, or else the second to leftmost and second to rightmost faces would cause the arrangement to have a higher c-value. Continuing this line of argument, we eventually see that $p_1 = p_2 = \ldots = p_k = 0$, contradicting our assumption that some s_i or p_i be nonzero. $\square$